Automated Construction of Examples in Algebraic Geometry

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Introduction

Topics in Algebraic Geometry:

making algebraic geometry more concrete

Goal: to create a searchable database of properties, theorems and examples in algebraic geometry, and answer questions like:

- *'Does there exist a scheme with property A but not property B?'*
- *'If a morphism has property A, does it also have property B?'*

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Initial (naive) idea

- database = lists of properties, examples and theorems
- property = string
- example = list of booleans
- theorem = list of assumptions + list of conclusions
- searching = iterating + blindly applying theorems

categoricalexamples.math.leidenuniv.nl

👉 link to the old version of website

What about?

- composition of morphisms
- fiber products of schemes
- other categories: rings, modules, topological spaces, ...
- functors: Spec, Γ, Hom, forget, ...
- families of examples: $\mathbb{A}^1_X$ for any X

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Canard 🦆

- Dependent type system (written in C++)
- By default `Prop : Type` and `Type : Type`
- Everything is a function

```
Function {
    name: String
    type: Function
    parameters: List Function
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Canard

Examples

```
let Ring : Type

let domain (R : Ring) : Prop

let domain_of_field {R : Ring} (h : field R) : domain R

let affine_line (X : Scheme) : Scheme

let zariski_local (P (X : Scheme) : Prop) : Prop
```

Canard 🦆

- Some functions are specializations

```
Specialization extends Function {
    base: Function
    arguments: List Function
}
```

- For example

```
let f (a b c : A) : B
def g (x : A) := f x x x
```

- Can construct expressions, axioms, theorems, examples, ...
- Theorems and examples are represented by the same type of object

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search (X : Scheme) (h1 : integral X) (h2 : affine X)
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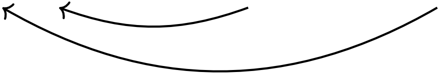

(1) Create ‘query’ (*telescope*)

(2) Resolve goals (starting at the back), by applying functions, creating new queries

(3) Continue recursively (breadth-first search) with some maximal depth, creating a tree of queries

(4) When we reach an empty query, do backsubstitution

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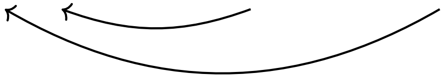

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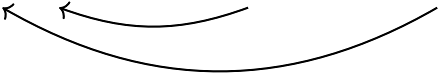

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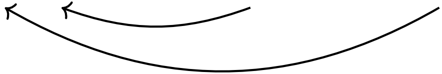

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-- (1) Start with this query
search (X : Scheme) (h1 : integral X) (h2 : affine X)
-- (2) Apply this theorem
spec_af (R : Ring) : affine (Spec R)
-- to get this query (remember X := Spec R, h2 := spec_af R)
search (R : Ring) (h1 : integral (Spec R))
-- (3) Apply this theorem
spec_int {R : Ring} (h : domain R) : integral (Spec R)
-- to get this query (remember h1 := spec_int h)
search (R : Ring) (h : domain R)
-- (4) Apply this theorem
ZZ_is_dm : domain ZZ
-- and done! (with R := ZZ and h := ZZ_is_dm)
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Note

- If goal has arguments, create ‘local context’ with variables

```
search (P (X : Scheme) : Prop)
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Optimizations

- Sort functions based on type before-hand
- Prioritize based on depth
- Cut-off unnecessary branches
- Multi-threading

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Now on to Lean(4)!

Aesop (by Jannis Limperg)

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-- (1) Mark functions with an attribute
@[aesop safe] theorem my_thm (P : Prop) : P := by { ... }
@[aesop unsafe 42%] axiom my_ax (R : Ring) : trivial R

-- (2) Use Aesop as tactic
theorem my_awesome_thm (R : Ring) : reduced R := by {
    aesop;
}

-- (3) possibly with extra theorems
example (R : Ring) : reduced R := by {
    aesop (add unsafe 10% my_awesome_thm);
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Aesop

How does Aesop search?

(1) apply safe rules

(2) apply unsafe rules, prioritize based on percentages

#query command

```
#query (X : Scheme) (h : X.affine) : (q : X.quasi_compact)
```

$\downarrow$

```
∀ (X : Scheme) (h : X.affine), ∃ (q : X.quasi_compact), True
```

$\downarrow$

```
call Aesop
```

$\downarrow$

```
extract objects and proofs
```

$\downarrow$

```
pretty print
```

TODO 📝

- Fix bugs / test cases
- User-friendly interface (website)
- Enlarge database
- Integrate with mathlib / swap definitions